The OLS estimator

Minimisse the sum of squard residuals

$$SSR(b_0, b_1) = \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = 0$$

The first order conditions are:

$$\frac{\partial SSR}{\partial \beta_0} = 2(-1)(y_1 - \hat{\beta}_0 - \hat{\beta}_1 x_1) + 2(-1)(y_2 - \hat{\beta}_0 - \hat{\beta}_1 x_2) + 2(-1)(y_3 - \hat{\beta}_0 - \hat{\beta}_1 x_3) + \cdots$$

$$= -2\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\frac{\partial SSR}{\partial \beta_1} = 2(-x_1)(y_1 - \hat{\beta}_0 - \hat{\beta}_1 x_1) + 2(-x_2)(y_2 - \hat{\beta}_0 - \hat{\beta}_1 x_2) + 2(-x_3)(y_3 - \hat{\beta}_0 - \hat{\beta}_1 x_3)$$

$$+ \cdots = -2\sum_{i=1}^{n} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

Or from a method of moments, a statistical approach

$$E(u) = 0 \text{ and } E(u|x) = 0$$

We get

$$\sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\sum_{i=1}^{n} x_i \hat{u}_i = \sum_{i=1}^{n} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

For $\hat{\beta}_0$

$$n\hat{\beta}_0 = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \hat{\beta}_1 x_i$$

$$\hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{i=1}^{n} \hat{\beta}_1 x_i$$

$$\boldsymbol{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}}$$

For $\hat{\beta}_1$

$$\sum_{i=1}^{n} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} \hat{\beta}_0 x_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = 0$$

$$\frac{1}{n}\sum_{i=1}^{n} y_i x_i - \hat{\beta}_0 \bar{x} - \hat{\beta}_1 \frac{1}{n}\sum_{i=1}^{n} x_i^2 = 0$$

Put $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ in to the equation

$$\frac{1}{n}\sum_{i=1}^{n} y_i x_i - \bar{y}\bar{x} = \hat{\beta}_1 (\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2)$$

$$\boldsymbol{\hat{\beta}_1 = \frac{\sum_{i=1}^{n} y_i x_i - n\bar{y}\bar{x}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

Note that $\hat{\beta}_1 = \frac{\widehat{Cov(x,y)}}{\widehat{Var(x)}} = \frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x^2}$